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A Darboux-Type Theorem for Slowly Varying Functions

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For functions $g(z)$ satisfying a slowly varying condition in the complex plane, we find asymptotics for the Taylor coefficients of the function

$$f(z) = g(z)\,(1-z)^{-\alpha}$$

when $\alpha > 0$. As applications we find asymptotics for the number of permutations with cycle lengths all lying in a given set S, and for the number having unique cycle lengths.

1. INTRODUCTION

Often the asymptotics of a function's Taylor coefficients can be determined by the behavior of the function near its singularities of smallest modulus. Information of these asymptotics is useful in probability, combinatorics and theoretical computer science. We briefly state three theorems that obtain such asymptotic information before presenting our results. More detailed information on all three is given in [6].

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The first result is sometimes known as Darboux's theorem. It states that if

$$f(z) = g(z)\,(1-z)^{-\alpha},$$

for $\alpha \notin \{0, -1, -2, -3, ...\}$, and if $g(z)$ has a radius of convergence larger than 1, then the Taylor coefficients f_n of $f(z) = \sum f_n z^n$ have asymptotic behavior

$$f_n \sim g(1)\, n^{\alpha-1}/\Gamma(\alpha)$$

To apply Darboux's theorem, the function $f(z)$ must therefore be analytically continuable across its circle of convergence.

The Tauberian theorem of Hardy, Littlewood and Karamata, described in [?], implies asymptotics for the Taylor coefficients of functions satisfying

$$f(z) = h((1-z)^{-1})\,(1-z)^{-\alpha}, \tag{1}$$

where $h(z)$ is a slowly varying function and $\alpha \geqslant 0$. A function $h(x)$ defined on $(0, \infty)$ and never 0 is said to be slowly varying if

$$\lim_{x \to \infty} h(\lambda x)/h(x) = 1 \tag{2}$$

for all $\lambda > 0$. Under these conditions

$$\sum_{k=0}^{n} f_k \sim n^\alpha h(n)/\Gamma(\alpha).$$

If in addition $\alpha > 0$ and the Taylor coefficients f_n are positive and monotonic for large n, then

$$f_n \sim n^{\alpha-1} h(n)/\Gamma(\alpha). \tag{3}$$

The condition of monotonicity on f_n can be difficult to check in practice.

The singularity analysis of Flajolet and Odlyzko in [4] gets results when $\alpha \notin \{0, -1, -2, -3, ...\}$. Their results imply the asymptotics (3) for functions satisfying (1) where $z \to 1$ within a domain in the complex plane of a certain sort that does not intersect the segment $[1, \infty)$. Their slowly varying condition on $h(z)$ when restricted to the real line amounts to

$$\lim_{x \to \infty} h(x \log^2 x)/h(x) = 1. \tag{4}$$

Singularity analysis theorems require the function $f(z)$ to be continuable across its circle of convergence when $\alpha \leqslant 1$.

Our main result, Theorem 2, considers functions of the form

$$f(z) = g(z)(1-z)^{-\alpha}$$

with $\alpha > 0$. Under certain conditions on $g(z)$, not including analytic continuation across its domain of convergence, the Taylor coefficients f_n of $f(z)$ have asymptotics

$$f_n \sim n^{\alpha-1} g(1-n^{-1})/\Gamma(\alpha).$$

In particular, we show that if $h(z)$ satisfies a slowly varying condition on the complex plane which restricted to the real axis gives the slowly varying condition (2), and not the more restrictive (4), and $g(z) = h((1-z)^{-1})$, then asymptotics (3) hold. We are able to get new results for the asymptotic number of permutations with cycle lengths restricted to a given set S. We also derive the asymptotics, first found by Greene and Knuth, for the number of permutations having unique cycle lengths. Our conditions on $g(z)$ are related to a definition of slowly varying functions due to Vuilleumier.

2. SLOWLY VARYING FUNCTIONS

The notion of slowly varying function we use is stated in Definition 1. It is essentially taken from [8], though we impose a weaker mode of convergence on $zh'(z)/h(z)$.

Definition 1. A function $h(z)$ is said to be slowly varying if $h(z)$ is analytic and non-zero in the half plane $H \equiv \{z : \operatorname{Re} z > 1/2\}$, and if

$$zh'(z)/h(z) \to 0$$

as $z \to \infty$ with in H.

Theorem 1 character characterizes slowly varying functions. In particular, functions which are slowly varying under Definition 1 satisfy condition (2) when restricted to the real line.

Theorem 1. *Let $h(z)$ be analytic and have no zeroes in the half plane H. The following two statements are equivalent*:

(1) *$h(z)$ is slowly varying.*

(2) *There exists a function $\sigma\colon (r, \infty) \to \mathbb{R}_+$ for some $r \geqslant 1/2$ such that σ is continuous, $\sigma(x) \downarrow 0$ as $x \to +\infty$, and such that for all $\lambda > 1$ and $z \in H$*

$$\left| \log \frac{h(\lambda z)}{h(z)} \right| \leqslant \sigma(|z|) \log \lambda. \tag{5}$$

In particular, $h(\lambda z)/h(z) \to 1$ as $z \to \infty$ within H for all $\lambda > 0$.

Moreover, (1) *and* (2) *imply that for all* $z_1, z_2 \in H$ *with* $|z_2| \geqslant |z_1|$

$$\left|\log \frac{h(z_2)}{h(z_1)}\right| \leqslant \sigma(|z_1|)\left(\log\left|\frac{z_2}{z_1}\right| + \left|\arg\left(\frac{z_2}{z_1}\right)\right|\right).$$

Proof. Defining $\omega(z)$ as

$$\omega(z) = zh'(z)/h(z),$$

and solving for $h(z)$ gives

$$h(z) = C\exp\left(\int_{z_0}^{z} \frac{\omega(u)}{u}\,du\right)$$

for some $z_0 \in H$ and $C \neq 0$.

Suppose that $h(z)$ is slowly varying and that $|z_2| > |z_1|$. Let $\sigma(x)$ be defined by

$$\sigma(x) = \sup_{\substack{y \in H \\ |y| \geqslant x}} |\omega(y)|.$$

From Definition 1, $\lim_{x\to\infty} \sigma(x) = 0$. We calculate

$$\begin{aligned}
\left|\log \frac{h(z_2)}{h(z_1)}\right| &= \left|\int_{z_1}^{z_2} \frac{\omega(u)}{u}\,du\right| \\
&\leqslant \sigma(|z_1|)\int_{z_1}^{z_2}\left|\frac{du}{u}\right| \\
&= \sigma(|z_1|)\int_{1}^{z_2/z_1}\left|\frac{du}{u}\right| \\
&\leqslant \sigma(|z_1|)\left(\int_1^{|z_2/z_1|}\frac{du}{u} + \int_{|z_2/z_1|}^{z_2/z_1}\left|\frac{du}{u}\right|\right) \\
&= \sigma(|z_1|)\left(\log\left|\frac{z_2}{z_1}\right| + \left|\arg\left(\frac{z_2}{z_1}\right)\right|\right),
\end{aligned}$$

where in the last integral we integrate over the circle $|u| = |z_2/z_1|$.

Conversely, if $h(z)$ satisfies (5), then

$$\begin{aligned}
|\omega(z)| &= \left|\frac{zh'(z)}{h(z)}\right| \\
&= \left|\frac{z}{h(z)}\lim_{r\downarrow 0}\frac{h((1+r)z) - h(z)}{rz}\right|
\end{aligned}$$

$$= \left| \lim_{r \downarrow 0} \frac{1}{r} \log \frac{h((1+r)\,z)}{h(z)} \right|$$

$$\leqslant \sigma(|z|) \lim_{r \downarrow 0} \frac{1}{r} \log(1+r)$$

$$= \sigma(|z|). \quad \blacksquare$$

The class of slowly varying functions includes $\log z$, $\log\log z$, and $\exp(\sqrt{\log z})$ for appropriate definitions of $\log z$, i.e. those that make the functions analytic on H. An example of a slowly varying function that does not satisfy (4) is $\exp(\log z/\log\log z)$.

Notice that a function $h(z)$ is slowly varying on the half plane H if and only if the function

$$g(z) = h((1-z)^{-1})$$

is analytic and non-zero on the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and

$$\frac{g'(z)(1-z)}{g(z)} \to 0$$

as $z \to 1$ within the unit disc.

Lemma 1 gets bounds needed for the proof of Theorem 2. Its proof is similar to that of Theorem 1 with $h(z) = g((1-z)^{-1})$.

LEMMA 1. *Given $0 < a < b < 2$ and $0 < \rho < 1$, let*

$$U(a,\, b,\, \rho) = \{z \in \mathbb{C} : a \leqslant |1-z| \leqslant b,\ |z| \leqslant \rho\}$$

and define

$$\tau = \sup_{z \in U(a,\, b,\, \rho)} \left| \frac{g'(z)(1-z)}{g(z)} \right|.$$

If $z_1, z_2 \in U(a, b, \rho)$ with $|1-z_1| \geqslant |1-z_2|$, and $\tau < \infty$, then

$$\left| \log \frac{g(z_2)}{g(z_1)} \right| \leqslant \tau \left(\log \left| \frac{1-z_1}{1-z_2} \right| + \left| \arg \frac{1-z_1}{1-z_2} \right| \right).$$

3. A DARBOUX-TYPE THEOREM

Theorem 2 is our main result; applications are given in Section 4. It can sometimes be applied even when $g'(z)$ is unbounded on $\{z \in \mathbb{C} : |z| < 1\}$.

THEOREM 2. *Let $g(z)$ be analytic on the disc $\{z \in \mathbb{C} : |z| < 1\}$ and suppose*

$$f(z) = g(z)(1-z)^{-\alpha}$$

for some $\alpha > 0$. Suppose there exists sequences ρ_n, θ_n in $\mathbb{R}_+$ such that $\rho_n < 1$, $n(1-\rho_n) \to 0$, and $n^{-1} < \theta_n < 2$, $n\theta_n \to \infty$, and with the following properties:

(1) *If $U_n = \{z \in \mathbb{C}: |z| \leqslant \rho_n,\ n^{-1} \leqslant |1-z| \leqslant \theta_n\}$, then*

$$\tau_n \equiv \sup_{z \in U_n} \left| \frac{g'(z)(1-z)}{g(z)} \right| \to 0 \qquad as \quad n \to \infty.$$

(2) *If Γ_n is the part of the circle $|z| = \rho_n$ where $|1-z| \geqslant \theta_n$, then*

$$\int_{\Gamma_n} |f'(z)\, dz| = f(1-n^{-1})\, o(1) \qquad as \quad n \to \infty.$$

If the assumptions are satisfied, then the nth Taylor coefficient f_n of f satisfies

$$f_n \sim g(1-n^{-1})\, n^{\alpha-1}/\Gamma(\alpha), \qquad n \to \infty.$$

Proof. Our proof uses estimates of Cauchy's integral formula. For any simple counter-clockwise curve γ about 0, the nth Taylor coefficient of an analytic function $f(z)$ is given by the integral

$$f_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z^{n+1}}\, dz. \tag{6}$$

From (6) comes the equation

$$\frac{f_n}{g(1-n^{-1})} = \frac{1}{2\pi i} \int_\gamma \frac{(1-z)^{-\alpha}}{z^{n+1}}\, dz + \frac{1}{2\pi i} \int_\gamma \left(\frac{g(z)}{g(1-n^{-1})} - 1 \right) \frac{(1-z)^{-\alpha}}{z^{n+1}}\, dz.$$

The first integral equals

$$\frac{1}{2\pi i} \int_\gamma \frac{(1-z)^{-\alpha}}{z^{n+1}}\, dz = (-1)^n \binom{-\alpha}{n} \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}.$$

For some $k > 0$ to be determined later we define the sequence

$$d_n = |1 - \rho_n e^{ik/n}|.$$

To prove the theorem it suffices to show that

$$\lim_{n\to\infty} n^{1-\alpha} \int_{\gamma(n)} \left(\frac{g(z)}{g(1-n^{-1})} - 1\right) \frac{(1-z)^{-\alpha}}{z^{n+1}}\, dz = 0, \tag{7}$$

where $\gamma(n)$ is the circle $\{z \in \mathbb{C} : |z| = \rho_n\}$ indented by the part of the circle $\{z : |z-1| = d_n\}$ in its interior, and where $\gamma(n)$ is traversed in the positive sense.

The limit (7) will be derived by bounding the integral on sub-paths of $\gamma(n)$. For a fixed integer $k>0$ to be determined later, the sub-paths are defined to be

$$\gamma_1(n) = \{z : |z-1| = d_n,\ -k/n \leqslant \arg z \leqslant k/n\},$$
$$\gamma_2(n) = \{z : |z| = \rho_n,\ k/n \leqslant \arg z \leqslant \theta_n\},$$

$$\gamma_3(n) = \{z : |z| = \rho_n,\ \theta_n \leqslant \arg z \leqslant 2\pi - \theta_n\},$$
$$\gamma_4(n) = \{z : |z| = \rho_n,\ 2\pi - \theta_n \leqslant \arg z \leqslant 2\pi - k/n\}.$$

We will show how to obtain (7) for the first three sub-paths only; the estimates for $\gamma_4(n)$ are similar to those for $\gamma_2(n)$.

We will need upper and lower bounds for $|1-\rho_n e^{it}|$ for $k/n \leqslant t \leqslant \pi$. Since $|1-e^{it}| = 2\sin(t/2) \in [2t/\pi, t]$, if $0 \leqslant t \leqslant \pi$ we have

$$|1-\rho_n e^{it}| \leqslant |1-e^{it}| + |e^{it}(1-\rho_n)| \leqslant t(1+t^{-1}(1-\rho_n))$$

and

$$|1-\rho_n e^{it}| \geqslant |1-e^{it}| - |e^{it}(1-\rho_n)| \geqslant t\left(\frac{2}{\pi} - t^{-1}(1-\rho_n)\right). \tag{8}$$

Here

$$\sup_{k/n \leqslant t \leqslant \pi} t^{-1}(1-\rho_n) \leqslant n(1-\rho_n)/k \to 0.$$

Bound on the Integral over γ_1

From (8) there is a positive constant C such that

$$d_n \geqslant Cn^{-1}.$$

We now estimate that

$$\sup_{z\in\gamma_1(n)} |z|^{-n-1} = (1-d_n)^{-n-1} = O(1)$$

and that

$$|1-z|^{-\alpha} \leqslant d_n^{-\alpha} = O(n^{\alpha}),$$

and bound the path length of $\gamma_1(n)$ by πn^{-1}. From Lemma 1, we have for $z \in \gamma_1(n)$ the bound

$$\left|\log \frac{g(z)}{g(1-n^{-1})}\right| \leqslant \tau_n(\log|n(1-z)| + |\arg(1-z)|).$$

Thus

$$\log \frac{g(z)}{g(1-n^{-1})} \to 0 \qquad \text{as} \quad n \to \infty$$

and therefore

$$\lim_{n\to\infty} n^{1-\alpha} \int_{\gamma_1(n)} \left(\frac{g(z)}{g(1-n^{-1})} - 1\right) \frac{(1-z)^{-\alpha}}{z^{n+1}}\, dz = 0.$$

Bound on the Integral over γ_2

It suffices to prove that given $\varepsilon > 0$, k can be chosen such that

$$\limsup_{n\to\infty} \left| n \int_{k/n}^{\theta_n} \frac{f(\rho_n e^{it})}{f(1-n^{-1})} (\rho_n e^{it})^{-n}\, i\, dt \right| < \varepsilon \tag{9}$$

and

$$\limsup_{n\to\infty} \left| n^{1-\alpha} \int_{k/n}^{\theta_n} (1-\rho_n e^{it})^{-\alpha} (\rho_n e^{it})^{-n}\, i\, dt \right| < \varepsilon.$$

As the function $g(z) \equiv 1$ is slowly varying, it is enough to show (9).

Integration by parts gives

$$\begin{aligned} n \int_{k/n}^{\theta_n} \frac{f(\rho_n e^{it})}{f(1-n^{-1})} (\rho_n e^{it})^{-n}\, i\, dt = &-\frac{f(\rho_n e^{i\theta_n})}{f(1-n^{-1})} (\rho_n e^{i\theta_n})^{-n} \\ &+ \frac{f(\rho_n e^{ik/n})}{f(1-n^{-1})} (\rho_n e^{ik/n})^{-n} \\ &+ \int_{k/n}^{\theta_n} \frac{f'(\rho_n e^{it})}{f(1-n^{-1})} (\rho_n e^{it})^{-n+1}\, i\, dt, \end{aligned}$$

where the derivative $f'(\rho_n e^{it})$ is taken with respect to $\rho_n e^{it}$. We will obtain estimates that imply (9) over γ_3.

Using Lemma 1 and (8), if $k/n \leqslant t \leqslant \theta_n$ we obtain

$$\left|\frac{f(\rho_n e^{it})}{f(1-n^{-1})}\right| = \left|\frac{g(\rho_n e^{it})}{g(1-n^{-1})}\,(n(1-\rho_n e^{it}))^{-\alpha}\right| \leqslant e^{\pi\tau_n}\left(nt\left(\frac{2}{\pi}-t^{-1}(1-\rho_n)\right)\right)^{-\alpha+\tau_n}. \tag{10}$$

Hence,

$$\lim_{n\to\infty}\left|\frac{f(\rho_n e^{i\theta_n})}{f(1-n^{-1})}\right| = 0$$

and

$$\limsup_{n\to\infty}\left|\frac{f(\rho_n e^{ik/n})}{f(1-n^{-1})}\right| = O(k^{-\alpha}). \tag{11}$$

For the third term, we estimate

$$\begin{aligned}\left|t\,\frac{f'(\rho_n e^{it})}{f(\rho_n e^{it})}\right| &= \left|t\left(\frac{g'(\rho_n e^{it})}{g(\rho_n e^{it})}+\frac{\alpha}{1-\rho_n e^{it}}\right)\right| \\ &\leqslant (\tau_n+\alpha)\,t\,|1-\rho_n e^{it}|^{-1} \\ &\leqslant (\tau_n+\alpha)\left(\frac{2}{\pi}-k^{-1}n(1-\rho_n)\right)^{-1}\end{aligned}$$

and using (10)

$$\begin{aligned}\limsup_{n\to\infty}\int_{k/n}^{\theta_n}\left|\frac{f'(\rho_n e^{it})}{f(1-n^{-1})}\right| i\,dt &= \limsup_{n\to\infty}\int_{k/n}^{\theta_n}\left|\frac{f'(\rho_n e^{it})}{f(\rho_n e^{it})}\right|\left|\frac{f(\rho_n e^{it})}{f(1-n^{-1})}\right| dt \\ &= \limsup_{n\to\infty} O(n^{-\alpha+\tau_n})\left|\int_{k/n}^{\theta_n} t^{-\alpha+\tau_n-1}\,dt\right| \\ &= \limsup_{n\to\infty} O(k^{-\alpha+\tau_n}) \\ &= O(k^{-\alpha}).\end{aligned} \tag{12}$$

By choosing k large enough we deduce (9) from (11) and (12).

Bound on the Integral over γ_3

Integrating by parts, we get

$$n\int_{\theta_n}^{2\pi-\theta_n}\frac{f(\rho_n e^{it})}{f(1-n^{-1})}(\rho_n e^{it})^{-n}\,i\,dt=-\frac{f(\rho_n e^{i(2\pi-\theta_n)})}{f(1-n^{-1})}(\rho_n e^{-i\theta_n})^{-n}$$
$$+\frac{f(\rho_n e^{i\theta_n})}{f(1-n^{-1})}(\rho_n e^{i\theta_n})^{-n}$$
$$-\int_{\theta_n}^{2\pi-\theta_n}\frac{f'(\rho_n e^{it})}{f(1-n^{-1})}(\rho_n e^{it})^{-n+1}\,i\,dt.$$

The first two terms on then right hand were bounded in the argument for γ_2, and the second condition of the theorem bounds the third term. Another argument using integration by parts and (8) that is omitted here shows

$$n^{1-\alpha}\int_{\gamma_3(n)}(1-z)^{-\alpha}z^{-n-1}\,dz=o(1)$$

and so (7) holds for $\gamma_3(n)$. This completes the proof of Theorem 2. ∎

4. APPLICATIONS

We present five corollaries of Theorem 2.

COROLLARY 1. *If $h(z)$ is a slowly varying function defined on H (cf. Def. 1) for which $h(z)$ and $\omega(z)=zh'(z)/h(z)$ are bounded in absolute value on H, then for $\alpha>0$ the Taylor coefficients f_n of the function*

$$f(z)=h((1-z)^{-1})(1-z)^{-\alpha}$$

have asymptotics

$$f_n\sim n^{\alpha-1}h(n)/\Gamma(\alpha).$$

Proof. If $\theta_n=n^{-\beta}$ for some $\beta\in(0,1)$ and ρ_n is any sequence such that $\rho_n\uparrow 1$ and $n(1-\rho_n)\to 0$, then the first condition of Theorem 2 is satisfied because $h(z)$ is slowly varying.

To show the second condition, we calculate

$$f'(z)=(\omega((1-z)^{-1})+\alpha)\,h((1-z)^{-1})(1-z)^{-\alpha-1},$$

implying

$$\frac{1}{f(1-n^{-1})}\int_{\Gamma_n}|f'(z)\,dz| = \frac{O(1)}{h(n)n^{\alpha}}\int_{n^{-\beta}}^{\pi} t^{-\alpha-1}\,dt$$
$$= \frac{O(1)}{h(n)n^{\alpha(1-\beta)}}.$$

Theorem 1 shows that $h(x)$ is slowly varying on the real line. A property of slowly varying functions $h(x)$ on the real line (see Proposition 1.3.6 on page 16 of [?], for example) is that for any $\kappa > 0$,

$$\lim_{n\to\infty} h(n)n^{\kappa} = \infty.$$

Hence, the second condition of Theorem 2 is satisfied. ∎

The next corollaries show that Theorem 2 can be used in some cases where $g'(z)$ is not bounded.

COROLLARY 2. *Let $g(z)$ be analytic and non-zero in $\Delta \equiv \{z \in \mathbb{C} : |z| < 1\}$ and suppose there exists a constant $q \in (0, 1)$ such that*

$$\sup_{|z|=\rho}\left|\frac{g'(z)}{g(z)}\right| = O(1)\,(1-\rho)^{-q} \qquad \text{as} \quad \rho \uparrow 1.$$

Let $f(z) = g(z)(1-z)^{-\alpha}$ where $\alpha > 0$. Under these conditions $g(z)$ approaches a limit μ as $z \to 1$ within Δ non-tangentially and the nth Taylor coefficient of f satisfies

$$f_n \sim \frac{\mu n^{\alpha-1}}{\Gamma(\alpha)}.$$

The condition that $z \to 1$ within Δ non-tangentially means that z is contained in a sector given by

$$|\arg(1-z)| \leqslant \frac{\pi}{2} - \varepsilon$$

for some $\varepsilon > 0$.

Proof. As $|z| \uparrow 1$ we have

$$\left|\log\frac{g(z)}{g(0)}\right| = O(1)\int_0^{|z|}(1-\rho)^{-q}\,d\rho$$
$$= O(1),$$

and it follows that $|g(z)|$ and $|g(z)|^{-1}$ are bounded on Δ. Therefore

$$g'(z) = O(1)(1-|z|)^{-q} \qquad \text{as} \quad |z| \uparrow 1$$

and $g(z)$ approaches a limit μ as $z \to 1$ non-tangentially in Δ.

We will apply Theorem 2 with $\rho_n = 1-(n \log n)^{-1}$ and $\theta_n = n^{-\beta}$ for some $\beta \in (q, 1)$ which satisfies $\beta > (q-\alpha)/(1-\alpha)$ if $\alpha < 1$. The first condition of Theorem 2 is satisfied because $\tau_n = O(1)(1-\rho_n)^{-q}\,\theta_n = o(1)$. To check the second condition, we use the estimate

$$\begin{aligned} f'(z) &= g'(z)(1-z)^{-\alpha} + \alpha g(z)(1-z)^{-\alpha-1} \\ &= O(1)(1-|z|)^{-q}\,(1-z)^{-\alpha} + O(1)(1-z)^{-\alpha-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{f(1-n^{-1})} \int_{\Gamma_n} |f'(z)\,dz| &= O(1)n^{-\alpha}((n \log n)^q\,(n^{-\beta(1-\alpha)} \log n + 1) + n^{\alpha\beta}) \\ &= o(1), \end{aligned}$$

since $q - \alpha - \beta(1-\alpha) < 0$, proving the second condition. ∎

In Corollary 3 we use a simple estimate to apply Corollary 2 to the number of permutations on n letters having unique cycle lengths. This example is treated in [5] by a somewhat complicated argument that involves verifying the conditions of a Tauberian theorem. A drawback of our method is that we do not obtain the higher order asymptotics found in [5].

The probability a random permutation on n letters has cycles of distinct lengths is given by the nth Taylor coefficient f_n of $f(z)$, where

$$f(z) = \prod_{i=1}^{\infty} \left(1 + \frac{z^i}{i}\right) \tag{13}$$

The number of permutations on n letters having distinct cycle lengths is then $n!f_n$. One may rewrite (13) as

$$f(z) = (1+z)\exp(\phi(z))(1-z)^{-1},$$

where $\phi(z)$ is defined as

$$\phi(z) = -z + \sum_{j=2}^{\infty} \frac{(-1)^{j-1}}{j} \sum_{k=2}^{\infty} \frac{z^{jk}}{k^j}.$$

COROLLARY 3. *The number u_n of permutations on n letters having unique cycle lengths satisfies*

$$u_n \sim e^{-\gamma} n!,$$

where γ is Euler's constant.

Proof. From the previous remarks $f_n = g_n + g_{n-1}$, where g_n is the nth Taylor coefficienty of

$$g(z) = \exp(\phi(z))(1-z)^{-1}.$$

It is not hard to show $\phi(1) = -\log 2 - \gamma$, where γ is Euler's constant. If $g(z) = \exp(\phi(z))$ satisfies the condition of Corollary 2, then the result will follow. We estimate

$$\begin{aligned}
\sup_{|z|=\rho} \left| \frac{g'(z)}{g(z)} \right| &= \sup_{|z|=\rho} |\phi'(z)| \\
&= 1 + \sup_{|z|=\rho} \left| \sum_{j=2}^{\infty} \sum_{k=2}^{\infty} \frac{(-1)^{j-1}}{k^{j-1}} z^{jk-1} \right| \\
&\leqslant 1 + \sup_{|z|=\rho} \left| \sum_{k=2}^{\infty} \frac{z^{2k-1}}{k} \right| + \sum_{k=2}^{\infty} \sum_{j=3}^{\infty} k^{1-j} \\
&= 1 + \sup_{|z|=\rho} \left| \frac{1}{z} \log \frac{1}{1-z^2} - z \right| + \sum_{k=2}^{\infty} \frac{1}{k(k-1)} \\
&= 2 + O\left(\log \frac{1}{1-\rho} \right) \\
&= O(1)(1-\rho)^{-q}
\end{aligned}$$

for any $q \in (0, 1)$. ▮

For $\alpha \geqslant 1$ weaker conditions may be used than in Corollary 2:

COROLLARY 4. *The conclusion of Corollary 2 remains valid if $\alpha \geqslant 1$ and if the condition on* $\sup |g'(z)/g(z)|$ *is replaced by*

$$\sup_{|z|=\rho} \left| \frac{g'(z)}{g(z)} \right| = O(1) \frac{1}{1-\rho} \left(\log \frac{1}{1-\rho} \right)^{-1-\varepsilon} \qquad \textit{as} \quad \rho \uparrow 1.$$

Proof. The existence of μ is shown as in the proof of Corollary 2. We prove the conditions of Theorem 2 are satisfied with $\rho_n = 1 - (n \log n)^{-1}$.

The first condition follows easily. As for the second condition of Theorem 2, letting

$$\psi(\rho) := \sup_{|z|=\rho} \left| \frac{g'(z)}{g(z)} \right|,$$

for $z \in \Delta$ we estimate

$$\begin{aligned} |f'(z)| &= |g'(z)(1-z)^{-\alpha} + \alpha g(z)(1-z)^{-\alpha-1}| \\ &= O(1)\, \psi(|z|)\, |1-z|^{-\alpha} + O(1)\, |1-z|^{-\alpha-1}, \end{aligned}$$

and therefore,

$$\begin{aligned} \frac{1}{f(1-n^{-1})} \int_{\Gamma_n} |f'(z)\, dz| &= O(1) n^{-\alpha} \int_{\theta_n}^{\pi} (\psi(\rho_n) + \theta^{-1}) \theta^{-\alpha}\, d\theta \\ &= O(1)(n\theta_n)^{-\alpha} (\psi(\rho_n)\theta_n \log(\theta_n^{-1}) + 1) \\ &= o(1). \quad \blacksquare \end{aligned}$$

Corollary 4 can be used to obtain the asymptotic behavior of the number of permutations having all cycle lengths restricted to some set $S \subset \mathbb{Z}_+$. Let $p_n(S)$ be the number of such permutations on n letters. If $f_n = p_n(S)/n!$ and $f(z) = \sum_n f_n z^n$, then with

$$g(z) = \exp\left(-\sum_{j \notin S} z^j/j \right)$$

we have

$$\begin{aligned} f(z) &= \prod_{j \in S} \exp(z^j/j) \\ &= (1-z)^{-1} \prod_{j \notin S} \exp(-z^j/j) \\ &= (1-z)^{-1} \exp\left(-\sum_{j \notin S} z^j/j \right) \\ &= (1-z)^{-1} g(z). \end{aligned}$$

Let $t(m) = |\{1, 2, 3, ..., m\} \backslash S|$. Bender [1] uses the Hardy, Littlewood and Karamata Tauberian theorem to show that if $t(m)/m \to 1 - \rho \in [0, 1]$ as $m \to \infty$, then

$$f_0 + f_1 + f_2 + \cdots + f_n \sim n g(1-n^{-1})/\Gamma(\rho+1).$$

The sequence f_n can be shown to be monotonic, and the second part of the Hardy, Littlewood and Karamata Tauberian applied, only for special cases.

We will get asymptotics for $p_n(S)$ when $t(m) = O(m(\log m)^{-2-\varepsilon})$ in Corollary 5. The remarks after the folowing theorem of Szegö [7] show that under this assumption on $t(m)$ the function $g(z)$ in general has no analytic continuation outside Δ. Hence Darboux's theorem and singularity analysis seem not to be applicable to this problem.

THEOREM 3. *If $f(z)$ has a Taylor expansion around* 0 *with only a finite number of different Taylor coefficients, then either the unit circle is the natural boundary of $f(z)$ (so f cannot be continued outside this circle), $f(z)$ is a polynomial divided by $(1-z^k)$ for some integer k, or $f(z)$ is a polynomial.*

Applying Theorem 3 to $g'(z)/g(z)$ shows that Δ is a natural boundary for $g(z)$ unless $g'(z)/g(z)$ is a polynomial or a polynomial divided by $(1-z^k)$ for some integer k. In the first case $\mathbb{Z}_+ \backslash S$ is finite and in the second $t(m)/m$ would converge to some positive number as $m \to \infty$, violating our assumption on the growth of $t(m)$.

COROLLARY 5. *If $S \subseteq Z^+$ is such that $t(m) = O(m(\log m)^{-2-\varepsilon})$ for some $\varepsilon > 0$, then the number of permutations with cycle lengths restricted to S is asymptotically*

$$p_n(S) \sim \exp\left(-\sum_{j \notin S} j^{-1}\right) n!$$

Proof. For any $m \geqslant 1$, the function $\psi(\rho)$ is bounded by

$$\begin{aligned}\psi(\rho) &\leqslant \sum_{j \notin S} \rho^{j-1} \\ &\leqslant t(m) + \sum_{j=m+1}^{\infty} \rho^{j-1} \\ &= t(m) + \frac{\rho^m}{1-\rho}.\end{aligned}$$

Choosing $m = m(\rho)$ for which

$$m \sim \frac{1}{1-\rho} \log \frac{1}{1-\rho},$$

we have

$$\psi(\rho) = O(1)\,\frac{1}{1-\rho}\left(\log\frac{1}{1-\rho}\right)^{-1-\varepsilon}.$$

Now apply Corollary 4. ∎

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